Name (IN CAPITALS): **Version #1**

Instructor and section number or class time: Shaun The Sheep, (break of dawn)

Math 10560 Exam 3
Apr. 19, 2022.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 16 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
11.	
12.	
13.	
14.	
Total	

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9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
Total	________

Multiple Choice

1.(6pts) Determine which **one** of the following series is divergent.

Solution:

(1) By the limit comparison test, we can compare this series to $\sum \frac{1}{n}$, which diverges and hence this series diverges.
(2) Geometric series with $a = 6/5$ $r = 3/5 < 1$, hence it converges.
(3) By the limit comparison test, we can compare this series to $\sum \frac{1}{n^2}$, which converges and hence this series converges.
(4) By the limit comparison test, we can compare this series to $\sum \frac{1}{n^{3/2}}$, which converges and hence this series converges.
(5) $\frac{1}{n}\left(\frac{2}{3}\right)^n < \left(\frac{2}{3}\right)^n$. Since the series $\sum \left(\frac{2}{3}\right)^n$ converges, by the comparison test $\sum \frac{1}{n}\left(\frac{2}{3}\right)^n$ converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^n$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}+n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2}{3}\right)^n$

2.(6pts) Consider the following series.

$$(I) \quad \sum_{n=1}^{\infty} \left(\frac{n+1}{4n-1}\right)^n \qquad\qquad (II) \quad \sum_{n=1}^{\infty} \frac{3^n}{(n-1)!}.$$

Which of the following is **true**?

Solution: We can start by ruling out the answer, "The ratio test is inconclusive on (II)."

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{3^{n+1}(n-1)!}{3^n n!}\right| &= \lim_{n\to\infty}\frac{3}{n}\\ &= 0\\ &< 1.\end{aligned}$$

Thus the ratio test implies (II) converges. This eliminates three possible answers.
It remains to check whether (I) converges or not. Because the terms of the series are taken to the n-th power, it looks like a good candidate for the root test.

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{n+1}{4n-1}\right)^{n\cdot\frac{1}{n}} &= \frac{1}{4}\\ &< 1.\end{aligned}$$

This implies (I) converges, so both series converge.

(a) Both of the series converge.

(b) (I) converges while (II) diverges.

(c) The ratio test is inconclusive on (II).

(d) (I) diverges while (II) converges.

(e) Both of the series diverge.

3.(6pts) Which one of the following series is conditionally convergent?

Solution: Convergers conditionally means the series must converge but the absolute series diverges.

(a) Using the alternate series test, with $b_n = \frac{n^2}{n^3+1}$.
$\lim_{n\to\infty} b_n = 0$, and

$$1 + \frac{1}{(n+1)^2} > \frac{1}{n^2} \Rightarrow n + 1 + \frac{1}{(n+1)^2} > n + \frac{1}{n^2}$$

and hence, $b_{n+1} < b_n$. The series converges.
The absolute series is $\sum \frac{n^2}{n^3+1}$, which by the limit comparison test, we can compare this to $\frac{1}{n}$ which converges and hence the series converges conditionally.

(b) The absolute series is $\sum(3/5)^n$ is a geometric series with $r < 1$, and hence the series absolutely converges.

(c) The absolute series is $\sum(1/\sqrt{n^5+1})$. By the limit comparison test we can compare this to $1/n^{5/2}$ which converges(p-series with $p > 1$), hence the series absolutely converges.

(d) The absolute series is $\sum(1/n^4)$ is a p-series with $p > 1$, and hence the series absolutely converges.

(e) $(-1)^n(3/2)^n$ divergers as $n \to \infty$, hence by the divergence test, the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^5+1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^n}$

4.(6pts) Which of the following statements are true about the series
$$\sum_{1}^{\infty} \frac{n^2+1}{n^5-n}?$$

Note (just in case you haven't encountered the verb) To deduce means to arrive at (a fact or a conclusion) by reasoning or to draw as a logical conclusion.

I. One can deduce that this series converges by observing that $\lim_{n\to\infty} \frac{n^2+1}{n^5-n} = 0$.

II. One can deduce that this series converges using the Ratio Test.

III. One can deduce that this series converges using the Limit Comparison Test, comparing with the p-series $\sum_{1}^{\infty} \frac{1}{n^3}$.

Solution: We quickly rule out I since this is not an alternating series.
If we attempt the ratio test, we see
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{((n+1)^2+1)(n^5-n)}{(n^2+1)((n+1)^5-(n+1))}\right| &= \lim_{n\to\infty}\left|\frac{n^7+\text{ lower degree terms}}{n^7+\text{ lower degree terms}}\right| \\ &= 1.\end{aligned}$$

This implies the ratio test is inconclusive, so we can rule out II. This only leaves two possible answers.
Checking III, we see
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(n^2+1)n^3}{n^5-n}\right| &= \lim_{n\to\infty}\left|\frac{n^5+n^3}{n^5-n}\right| \\ &= 1 \\ &< \infty.\end{aligned}$$

So the series converges by the limit comparison test, and we conclude only III is true.

(a) III only (b) II, III only (c) I, II only

(d) I, III only (e) None

5.(6pts) Find a power series representation for the function

$$\frac{2}{(1-x)^2}$$

in the interval $(-1, 1)$.
(Hint: Differentiating a well-known power series may help).

Solution: Consider the power series of $\frac{1}{1-x}$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Differentiating the above power series,

$$\begin{aligned} \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1} \\ \frac{2}{(1-x)^2} &= \sum_{n=1}^{\infty} 2nx^{n-1} \end{aligned}$$

(a) $\displaystyle\sum_{n=1}^{\infty} 2nx^{n-1}$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n 2nx^{n-1}$ (c) $\displaystyle\sum_{n=1}^{\infty} 2^n nx^{n-1}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{2x^{n+1}}{n}$ (e) $\displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n x^{n+1}}{n+1}$

6.(6pts) Which of the power series given below is a power series representation of the function

$$f(x) = x\cos(\sqrt{x})$$

centered at 0?

Solution: We use the power series representation for $\cos(x)$ to get

$$\begin{aligned} x\cos(\sqrt{x}) &= x\sum_{n=0}^{\infty}\frac{(-1)^n(x^{1/2})^{2n}}{(2n)!} \\ &= x\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(2n)!}. \end{aligned}$$

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(2n)!}$. (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$. (c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(2n+1)!}$.

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{(2n+1)!}$. (e) $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$.

7.(6pts) Consider the function $f(x)$ defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^{2n}}{2^n n!}, \qquad -\infty < x < \infty.$$

Which of the following statements is true?

Solution:

(a) Differentiating each term,

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^{2n-1}}{2^{n-1}(n-1)!}$$

(b) Integrating each term,

$$\int f(x)\, dx = C + \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^{2n+1}}{2^n n!(2n+1)}$$

(c) part(a) we have computed the derivative

(d) $f'(2) = 0$

(e) $f(2) = 1$

(a) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^{2n-1}}{2^{n-1}(n-1)!}$

(b) $\int f(x)\, dx = C + \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^{2n+1}}{2^n (2n+1)!}$

(c) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^{2n-1}}{2^n n!}$

(d) $f'(2) = -1$

(e) $f(2) = 0$

8.(6pts) Use a well known power series to find the sum of the following series

$$\sum_{n=0}^{\infty} \frac{3^n \pi^n}{n!}$$

Solution: We use the power series representation of e^x to see

$$\sum_{n=0}^{\infty} \frac{3^n \pi^n}{n!} = \sum_{n=0}^{\infty} \frac{(3\pi)^n}{n!}$$
$$= e^{3\pi}.$$

(a) $e^{3\pi}$ (b) 1 (c) $\cos 3$ (d) -1 (e) 0

9.(6pts) The degree 3 Taylor polynomial of

$$f(x) = \ln(x)$$

centered at $a = 2$ is given by:

Solution:

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3,$$

Now,

$$f(x) = \ln x,\ f'(x) = \frac{1}{x},\ f''(x) = -\frac{-1}{x^2},\ f'''(2) = \frac{2}{x^3}.$$

$$f(2) = \ln 2,\ f'(2) = \frac{1}{2},\ f''(2) = -\frac{1}{2^2},\ f'''(2) = \frac{1}{2^2}.$$

Hence, the Taylor polynomial of dergee 3 is,

$$T_3(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3.$$

$$T_3(x) = \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24}.$$

(a) $T_3(x) = \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24}$

(b) $T_3(x) = \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{4} + \frac{(x-2)^3}{4}$

(c) $T_3(x) = 1 + \frac{x-2}{2} - \frac{(x-2)^2}{4} + \frac{(x-2)^3}{4}$

(d) $T_3(x) = 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24}$

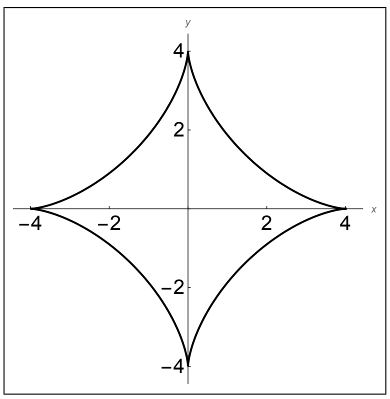
(e) $T_3(x) = \ln(2) + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{4}$

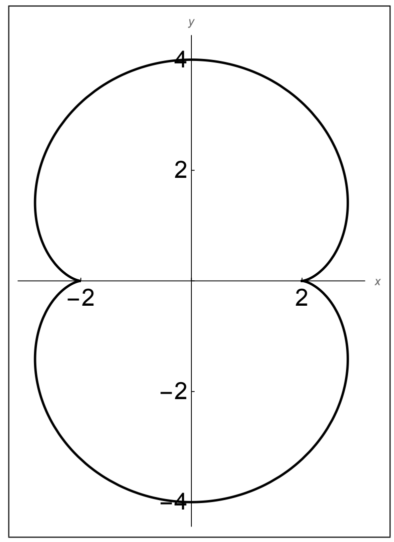
10.(6pts) Which of the following is a graph of the parametric curve defined by

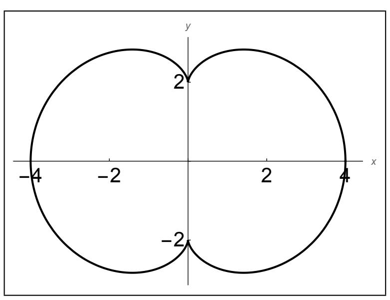
$$x = 3\cos(t) + \cos(3t), \qquad y = 3\sin(t) - \sin(3t)$$

for $0 \le t \le 2\pi$?

Solution: We plug values for t into the equations for x and y to eliminate possible answers.
Plugging in 0 for t gives the point $(4, 0)$, so we can rule out (b).
Plugging in $\pi/2$ for t gives the point $(0, 4)$, so we can rule out (c) and (e).
Plugging in $3\pi/2$ for t gives the point $(0, -4)$, so we conclude the correct answer is (a).

(a)


(b)


(c)


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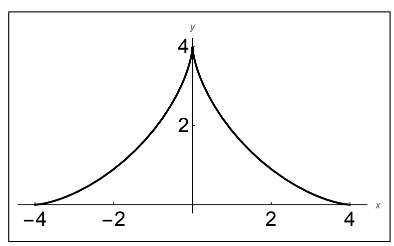
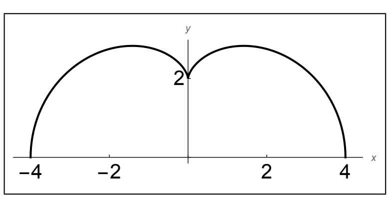


(e)


Partial Credit

Please show all of your work for credit in questions 11-13.
If some of the work that you wish to have considered for points is on another page, please indicate where the work is with words and arrows.

11.(13pts) Find the radius of convergence and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{3^n (x+2)^n}{n}.$$

If, in the course of the solution, you test for convergence of a series, please state clearly which test you are using.

Solution: Let, $a_n = \dfrac{3^n(x+2)^n}{n}$. By the ratio test we want

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Which implies,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{3^{n+1}|x+2|^{n+1}}{n+1}}{\frac{3^n|x+2|^n}{n}} = 3|x+2| \lim_{n\to\infty} \frac{n}{n+1} < 1$$

Since, $\lim_{n\to\infty} \frac{n}{n+1} = 1$, we get

$$|x+2| < \frac{1}{3}.$$

Hence we get that **the radius of convergence is** $\frac{1}{3}$
Note

$$|x+2| < \frac{1}{3} \Rightarrow -\frac{1}{3} < x+2 < \frac{1}{3}$$

Hence, $-\frac{7}{3} < x < -\frac{5}{3}$ By the Ratio test the series converges for $x \in (-7/3, -5/3)$, and diverges for $x \in (-\infty, -7/3) \cup (-5/3, \infty)$. So we are left with to check for convergence at the end points.
$x = -\frac{5}{3}$ the series is

$$\sum_{n=1}^{\infty} \frac{3^n(-5/3+2)^n}{n} = \sum_{n=1}^{\infty} \frac{3^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

The series diverges as it is a p series with $p = 1$
$x = -\frac{7}{3}$ the series is,

$$\sum_{n=1}^{\infty} \frac{3^n(-7/3+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n(-1)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This series converges by the Alternating Series test, as $\lim_{n\to\infty} \frac{1}{n} = 0$, and $\frac{1}{n}$ is a decreasing function on n.
Hence the **Interval of convergence is** $[-7/3, -5/3)$

12.(13pts) (a) Find a power series representation (with center 0) for the antiderivative

$$F(x) = \int \frac{1}{1+x^7}\,dx,$$

which satisfies the initial condition: $F(0) = 0$.
Hint: Use your knowledge of a well known series.

Solution: We use the power series representation for $1/(1-x)$ to see

$$\begin{aligned} F(x) &= \int \frac{dx}{1-(-x^7)} \\ &= \int \sum_{n=0}^{\infty} (-x^7)^n dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+1}}{7n+1} + C. \end{aligned}$$

The initial condition $F(0) = 0$ implies $C = 0$.
(b) Use part (a) to find an expression for the definite integral

$$\int_0^1 \frac{1}{1+x^7}\,dx,$$

as the sum of an infinite series (note: the variable x should not appear in your answer)

Solution: Using part (a), we get

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^7} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+1}}{7n+1} \Bigg|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{7n+1}}{7n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{7n+1}}{7n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1}. \end{aligned}$$

(c) Use the alternating series estimation theorem to estimate the value of the above definite integral (in part (b)) so that the error of estimation is less than $\dfrac{1}{10}$.
Solution: Writing out the first few terms, we see

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1} = 1 - \frac{1}{8} + \frac{1}{15} - \frac{1}{22} + \cdots.$$

Since $\frac{1}{8} \le \frac{1}{10} \le \frac{1}{15}$, we have

$$\int_0^1 \frac{dx}{1+x^7} \approx 1 - \frac{1}{8} = \frac{7}{8}$$

with error $\le \frac{1}{15} < \frac{1}{10}$.

13.(13pts) For parts (a) and (b) below, consider the parametric curve defined by

$$x = 1 + \cos(3t) \qquad y = \sin(3t)$$

for $0 \leq t \leq \dfrac{\pi}{3}$.

(a) Find the arclength of the given curve.
(a formula from the formula sheet should help.)
Solution: The arc length is given by,

$$l = \int_0^{\frac{\pi}{3}} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

Computing the derivatives, $\frac{dy}{dt} = -3\sin(3t)$, $\frac{dy}{dt} = 3\cos(3t)$. We get arc length,

$$l = \int_0^{\frac{\pi}{3}} \sqrt{9\sin^2(3t) + 9\cos^2(3t)} dt = \int_0^{\frac{\pi}{3}} 3dt = \pi$$

(b) Find the equation of the tangent line to the curve at the point on the curve where $t = \pi/6$.
Solution: We need to compute the slope. Now,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos(3t)}{-3\sin(3t)} = -\cot(3t)$$

The slope at $t = \frac{\pi}{6}$, is $\cot(\frac{\pi}{2}) = 0$. // We also need to know the point at whioch we are computing the tangent, for $t = \frac{\pi}{6}$,

$$(x, y) = (1 + \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2})) = (1, 1).$$

Hence the tangent line is given by $y - 1 = 0(x - 1)$, or

$$y = 1$$

14.(1pts) You will be awarded this point if you write your section number or class time next to the name of your instructor and you mark your answers on the front page with an X (not an O) . You may also use this page for

ROUGH WORK

16. Initials: ____________

The following is the list of useful trigonometric formulas:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$